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On the additional invariance of arbitrary spin relativistic wave equations

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Abstract. We demonstrate that the recently derived relativistic Schrödinger wave equations for particles of arbitrary spin and non-vanishing mass also exhibit additional invariance originally pointed out by Fushchich in the case of the Dirac and Maxwell equations.

In a recent paper, Fushchich (1974) has shown that the Dirac and Maxwell equations are invariant under a new set of operators different from the ones satisfying the Lie algebra of the Poincaré group. The purpose of the present note is to point out that such an additional invariance exists also for the recently discussed (Weaver *et al* 1964, Mathews 1966a,b, 1967a,b, Seetharaman *et al* 1971, Jayaraman 1973a,b) Schrödinger types of wave equations describing massive particles of arbitrary spin.

In Mathews' formalism of particles of arbitrary spin s and mass m, the Schrödinger equations

$$\mathrm{i}\frac{\partial\psi}{\partial t}=H\psi, \qquad \psi=\begin{pmatrix}\psi^{(0,s)}\\ \psi^{(s,0)}\end{pmatrix} \tag{1}$$

with ψ transforming locally according to the $2(2s+1)$-dimensional representation $D(0, s)\oplus D(s, 0)$ of the homogeneous Lorentz group, are invariant under the operations of the Poincaré group. The generators of this in the space of wavefunctions ψ are

$$P_0=p_0\equiv-\mathrm{i}\frac{\partial}{\partial t}=-H \tag{2a}$$

$$\boldsymbol{P}=\boldsymbol{p}\equiv-\mathrm{i}\boldsymbol{\nabla} \tag{2b}$$

$$\boldsymbol{J}=\boldsymbol{x}\times\boldsymbol{p}+\boldsymbol{S}, \qquad \boldsymbol{S}=\begin{pmatrix}\boldsymbol{s} & 0\\ 0 & \boldsymbol{s}\end{pmatrix} \tag{2c}$$

$$\boldsymbol{K}=t\boldsymbol{p}+\boldsymbol{x}p_0+\mathrm{i}\boldsymbol{\lambda}=t\boldsymbol{p}-\boldsymbol{x}H+\mathrm{i}\boldsymbol{\lambda}, \qquad \boldsymbol{\lambda}=\rho_3\boldsymbol{S}=\begin{pmatrix}\boldsymbol{s} & 0\\ 0 & -\boldsymbol{s}\end{pmatrix}. \tag{2d}$$

Here the matrices $(s_1, s_2, s_3)=\boldsymbol{s}$ are a $(2s+1)$-dimensional representation of the angular momentum operators and $\rho_i (i=1,2,3)$ are the $2(2s+1)$-dimensional Pauli matrices.

The Hamiltonians $H=H_{\mathrm{I}}(H_{\mathrm{II}})$ appropriate for half-integer(integer) spin particles are (Mathews 1967a,b)

$$H_{\mathrm{I}}=E\Big(\sum_\nu \tanh(2\nu\theta)C_\nu+\rho_1\sum_\nu \operatorname{sech}(2\nu\theta)B_\nu\Big) \tag{3a}$$

and

$$H_{\rm II}=E\Big(\sum_\nu \coth(2\nu\theta)C_\nu+\rho_1\sum_\nu \operatorname{cosech}(2\nu\theta)C_\nu)\Big). \tag{3b}$$

In (3*a*) and (3*b*)

$$E=m\cosh\theta, \qquad p=m\sinh\theta \tag{4}$$

and B_ν and C_ν are the combinations

$$B_\nu=\Lambda_\nu+\Lambda_{-\nu}, \qquad C_\nu=\Lambda_\nu-\Lambda_{-\nu} \tag{5a}$$

of the projection operators Λ_ν to the eigenvalue ν of $\lambda_p=(\boldsymbol{\lambda}\,.\,\boldsymbol{p})/p$ and satisfy

$$B_\mu B_\nu=C_\mu C_\nu=B_\mu\delta_{\mu\nu}, \qquad B_\mu C_\nu=C_\mu\delta_{\mu\nu}. \tag{5b}$$

We shall now proceed to prove our assertion that equations (1) with $H=H_{\rm I}(H_{\rm II})$ for half-integer (integer) spins are invariant with respect to a set of operators Q such that

$$\left[\mathrm{i}\frac{\partial}{\partial t}-H, Q\right]_-\psi=0 \tag{6}$$

and give a prescription to find the operators Q. To do this one invokes the theorem (Johnson and Chang 1974) that if A and B are two operators such that $A^2=B^2=1$ and which anticommute, $[A, B]_+=0$, then the transformation operators

$$U=\frac{1}{\sqrt{2}}(1+AB), \qquad U^{-1}=\frac{1}{\sqrt{2}}(1-AB) \tag{7}$$

achieve

$$UBU^{-1}=A. \tag{8}$$

Identifying B with $H_{\rm I}/E(H_{\rm II}/E)$ one has a choice of $A=\rho_2(\rho_1)$ so that

$$U_{\rm I}\frac{H_{\rm I}}{E}U_{\rm I}^{-1}=\frac{1}{\sqrt{2}}\left(1+\rho_2\frac{H_{\rm I}}{E}\right)\frac{H_{\rm I}}{E}\frac{1}{\sqrt{2}}\left(1-\rho_2\frac{H_{\rm I}}{E}\right)=\rho_2 \tag{9a}$$

and

$$U_{\rm II}\frac{H_{\rm II}}{E}U_{\rm II}^{-1}=\frac{1}{\sqrt{2}}\left(1+\rho_1\frac{H_{\rm II}}{E}\right)\frac{H_{\rm II}}{E}\frac{1}{\sqrt{2}}\left(1-\rho_1\frac{H_{\rm II}}{E}\right)=\rho_1. \tag{9b}$$

It is trivial to check that $U_{\rm I}$ is unitary

$$U_{\rm I}^{-1}=U_{\rm I}^\dagger \tag{10a}$$

as $H_{\rm I}$ and ρ_2 are Hermitian. However $U_{\rm II}$ is pseudo-unitary in the sense that

$$U_{\rm II}^{-1}=\rho_2U_{\rm II}^\dagger\rho_2 \tag{10b}$$

as $H_{\rm II}$ is pseudo-Hermitian, $H_{\rm II}^\dagger=\rho_3H_{\rm II}\rho_3$.

If now one defines a new representation χ by

$$\psi\to\chi=U\psi \tag{11}$$

with $U=U_{\rm I}(U_{\rm II})$ for half-integer(integer) spin cases, the Poincaré generators (2*a*–*d*) in the ψ representation are transformed in the χ representation into

$$P_{0\chi} = p_0 \equiv -\mathrm{i}\frac{\partial}{\partial t} = -\mathscr{H} = \begin{cases} -\rho_2 E & \text{for } U_{\mathrm{I}} \\ -\rho_1 E & \text{for } U_{\mathrm{II}} \end{cases} \tag{12a}$$

$$\boldsymbol{P}_\chi = \boldsymbol{p} \equiv -\mathrm{i}\boldsymbol{\nabla} \tag{12b}$$

$$\boldsymbol{J}_\chi = \boldsymbol{x} \times \boldsymbol{p} + \boldsymbol{S} \tag{12c}$$

$$\boldsymbol{K}_\chi = \begin{cases} t\boldsymbol{p} - \boldsymbol{x}\rho_2 E - \rho_1 \boldsymbol{S}\dfrac{H_{\mathrm{I}}}{E} & \text{for } U_{\mathrm{I}} \\ t\boldsymbol{p} - \boldsymbol{x}\rho_1 E + \rho_2 \boldsymbol{S}\dfrac{H_{\mathrm{II}}}{E} & \text{for } U_{\mathrm{II}}. \end{cases} \tag{12d}$$

One easily verifies in the χ representation that

$$\left[\mathrm{i}\frac{\partial}{\partial t} - \mathscr{H}, Q_\chi\right]_- \chi = 0 \tag{13}$$

with $\mathscr{H} = \mathscr{H}_{\mathrm{I}}(\mathscr{H}_{\mathrm{II}}) = \rho_2 E(\rho_1 E)$ for half-integer(integer) spin cases and the set of operators Q_χ being given.by

$$\tilde{P}_{0\chi} = -\mathscr{H} = \begin{cases} -\rho_2 E & \text{for } U_{\mathrm{I}} \\ -\rho_1 E & \text{for } U_{\mathrm{II}} \end{cases} \tag{14a}$$

$$\tilde{\boldsymbol{P}}_\chi = -\mathrm{i}\boldsymbol{\nabla} \tag{14b}$$

$$\tilde{\boldsymbol{J}}_\chi = \boldsymbol{x} \times \boldsymbol{p} + \boldsymbol{S} \tag{14c}$$

$$\tilde{\boldsymbol{K}}_\chi = \begin{cases} t\boldsymbol{p} - \dfrac{\rho_2}{2}(\boldsymbol{x}E + E\boldsymbol{x}) & \text{for } U_{\mathrm{I}} \\ t\boldsymbol{p} - \dfrac{\rho_1}{2}(\boldsymbol{x}E + E\boldsymbol{x}) & \text{for } U_{\mathrm{II}}. \end{cases} \tag{14d}$$

It is straightforward though tedious to find the set of operators

$$Q = U^{-1} Q_\chi U \tag{15}$$

under which equation (1) with the Hamiltonian $H_{\mathrm{I}}(H_{\mathrm{II}})$ is invariant and hence satisfies (6). Equation (15) leads to the following explicit expressions for the set of operators Q:

$$\tilde{P}_0 = -H_{\mathrm{I}} \quad \text{for } U_{\mathrm{I}}, \qquad \tilde{P}_0 = -H_{\mathrm{II}} \quad \text{for } U_{\mathrm{II}} \tag{16a}$$

$$\tilde{\boldsymbol{P}} = -\mathrm{i}\boldsymbol{\nabla} \tag{16b}$$

$$\tilde{\boldsymbol{J}} = \boldsymbol{x} \times \boldsymbol{p} + \boldsymbol{S} \tag{16c}$$

$$\tilde{\boldsymbol{K}} = \begin{cases} t\boldsymbol{p} - \dfrac{1}{2}[\boldsymbol{x}, H_{\mathrm{I}}]_+ + \dfrac{\mathrm{i}\rho_2}{2E}[H_{\mathrm{I}}, \boldsymbol{\lambda}]_- \\ t\boldsymbol{p} - \dfrac{1}{2}[\boldsymbol{x}, H_{\mathrm{II}}]_+ + \dfrac{\mathrm{i}\rho_1}{2E}[H_{\mathrm{II}}, \boldsymbol{\lambda}]_-. \end{cases} \tag{16d}$$

The set of operators Q_χ and hence Q as well, by virtue of (15), satisfy the following commutation relations (in writing which uniformly for both sets the subscript χ is suppressed)

$$\begin{aligned}
&[\tilde{P}_0, \tilde{P}_i]_- = 0, \qquad [\tilde{P}_i, \tilde{P}_j]_- = 0, \qquad [\tilde{K}_i, \tilde{P}_j]_- = \mathrm{i}\delta_{ij}\tilde{P}_0, \\
&[\tilde{J}_i, \tilde{P}_j]_- = \mathrm{i}\epsilon_{ijk}\tilde{P}_k, \qquad [\tilde{J}_i, \tilde{J}_j]_- = \mathrm{i}\epsilon_{ijk}\tilde{J}_k, \\
&[\tilde{K}_i, \tilde{K}_j]_- = -\mathrm{i}\epsilon_{ijk}(\tilde{J}_k - \tilde{S}_k), \\
&[\tilde{J}_i, \tilde{K}_j]_- = \mathrm{i}\epsilon_{ijk}\tilde{K}_k, \qquad (i, j, k = 1, 2, 3),
\end{aligned} \tag{17}$$

which are different from the commutation relations of the Poincaré generators.

In (17)

$$\tilde{\boldsymbol{S}}_\chi = \boldsymbol{S} \tag{18}$$

and $\mathscr{H} = \mathscr{H}_{\mathrm{I}}(\mathscr{H}_{\mathrm{II}})$ commutes with $\boldsymbol{S}$ whence one obtains

$$\left[\mathrm{i}\frac{\partial}{\partial t} - \mathscr{H}, \boldsymbol{S}\right]_- \chi = 0 \tag{19}$$

and as well

$$U^{-1}\left[\mathrm{i}\frac{\partial}{\partial t} - \mathscr{H}, \boldsymbol{S}\right]_- U(U^{-1}\chi) = \left[\mathrm{i}\frac{\partial}{\partial t} - H, \tilde{\boldsymbol{S}}\right]_- \psi = 0. \tag{20}$$

It is not hard to find that the explicit form of $\tilde{\boldsymbol{S}}$ is

$$\tilde{\boldsymbol{S}} = U^{-1}\boldsymbol{S}U = \begin{cases} \boldsymbol{S} - \dfrac{1}{2}\left(\rho_2 + \dfrac{H_{\mathrm{I}}}{E}\right)\left[\dfrac{H_{\mathrm{I}}}{E}, \boldsymbol{S}\right]_- & \text{(for half-integer spin case)}, \quad (21a) \\[2ex] \boldsymbol{S} - \dfrac{1}{2}\left(\rho_1 + \dfrac{H_{\mathrm{II}}}{E}\right)\left[\dfrac{H_{\mathrm{II}}}{E}, \boldsymbol{S}\right]_- & \text{(for integer spin case)}. \quad (21b) \end{cases}$$

One can evaluate the commutators $[(H/E), \boldsymbol{S}]_-$ in (21) by making use of the relevant formulae given in the appendices of papers by Seetharaman *et al* (1971) or Jayaraman (1973a) but is not attempted here. It follows immediately that when the generators $\tilde{\boldsymbol{S}}$ of (21) are added to Q of (16) the set of generators $\{\tilde{P}_0, \tilde{\boldsymbol{P}}, \tilde{\boldsymbol{J}}, \tilde{\boldsymbol{K}}, \tilde{\boldsymbol{S}}\}$ forms a Lie algebra under which the relativistic Schrödinger equation (1) is invariant for any spin with the understanding that $H = H_{\mathrm{I}}(H_{\mathrm{II}})$ is employed for half-integer(integer) spins.

Fushchich (1974) has also pointed out that the Dirac Hamiltonian commutes with a given set of generators satisfying the Lie algebra of the group O_4, which are however non-local in configuration space. We present a brief discussion below showing that such a property is also true of the relativistic Hamiltonians (2*a*,*b*) under consideration.

If one adds to the generators $\tilde{\boldsymbol{S}}_\chi = \boldsymbol{S}$ of (18) three more generators $\tilde{\boldsymbol{M}}_\chi = \rho_2\boldsymbol{S}(\rho_1\boldsymbol{S})$ for half-integer(integer) spin cases, then the generators

$$\begin{aligned}
&\tilde{S}^\chi_{\mu\nu} = (S^\chi_{23}, S^\chi_{31}, S^\chi_{12}, S^\chi_{14}, S^\chi_{24}, S^\chi_{34}) = (\tilde{\boldsymbol{S}}_\chi, \tilde{\boldsymbol{M}}_\chi) \\
&\tilde{S}^\chi_{\mu\nu} = -\tilde{S}^\chi_{\nu\mu} \qquad (\mu, \nu = 1, 2, 3, 4)
\end{aligned} \tag{22}$$

satisfy the Lie algebra

$$\begin{aligned}
&[\tilde{S}_{i\chi}, \tilde{S}_{j\chi}]_- = \mathrm{i}\epsilon_{ijk}\tilde{S}_{k\chi}, \\
&[\tilde{M}_{i\chi}, \tilde{S}_{j\chi}]_- = \mathrm{i}\epsilon_{ijk}\tilde{M}_{k\chi}, \\
&[\tilde{M}_{i\chi}, \tilde{M}_{j\chi}]_- = \mathrm{i}\epsilon_{ijk}\tilde{S}_{k\chi}, \qquad (i, j, k = 1, 2, 3),
\end{aligned} \tag{23}$$

of O_4 and $\mathscr{H}=\mathscr{H}_{\mathrm{I}}(\mathscr{H}_{\mathrm{II}})$ commutes with these generators:

$$[\tilde{S}^{\chi}_{\mu\nu}, \mathscr{H}]_{-}=0. \tag{24}$$

From (24) together with

$$\tilde{S}_{\mu\nu}=U^{-1}\tilde{S}^{\chi}_{\mu\nu}U$$

and

$$H=U^{-1}\mathscr{H}U$$

one finds that

$$[\tilde{S}_{\mu\nu}, H]_{-}=0 \tag{25}$$

where

$$\tilde{S}_{\mu\nu}=(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{M}}). \tag{26}$$

The generators $\boldsymbol{S}$ in (26) are given by (21*a*,*b*) respectively for the half-integer and integer spins while

$$\tilde{\boldsymbol{M}}=\begin{cases} \rho_2\boldsymbol{S}-\dfrac{1}{2}\left(\rho_2+\dfrac{H_{\mathrm{I}}}{E}\right)\left[\dfrac{H_{\mathrm{I}}}{E}, \rho_2\boldsymbol{S}\right]_{-} & \text{(for half-integer spin case)}, \quad (27a)\\ \rho_1\boldsymbol{S}-\dfrac{1}{2}\left(\rho_1+\dfrac{H_{\mathrm{II}}}{E}\right)\left[\dfrac{H_{\mathrm{II}}}{E}, \rho_1\boldsymbol{S}\right]_{-} & \text{(for integer spin case)}. \quad (27b)\end{cases}$$

Evidently the generators $\tilde{S}_{\mu\nu}$ satisfy the Lie algebra of O_4 and are non-local in configuration space. Equation (25) testifies to the O_4 symmetry of the relativistic Hamiltonians $H_{\mathrm{I}}(H_{\mathrm{II}})$ for any spin. Our proof complements Fushchich's (1974) assertion of an O_4 invariance for free particle equations for any spin in the canonical form.

The unitary (pseudo-unitary) operators discussed here and in a recent note (Jayaraman 1975) have an important bearing on the possibilities of constructing linear relativistic wave equations for any spin without manifest covariance and local covariance, the details of which will form the subject matter for a future publication.

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References

Fushchich V I 1974 *Lett. Nuovo Cim.* **11** 508–12
Jayaraman J 1973a *Nuovo Cim.* A **13** 877–96
—— 1973b *Nuovo Cim.* A **14** 343–62
—— 1975 *J. Phys. A: Math. Gen.* **8** L1–4
Johnson J E and Chang K K 1974 *Phys. Rev.* D **10** 2421–7
Mathews P M 1966a *Phys. Rev.* **143** 978–85
—— 1966b *Phys. Rev.* **143** 985–9
—— 1967a *Phys. Rev.* **155** 1415–20
—— 1967b *J. Math. Phys. Sci.* **1** 197–206
Seetharaman M, Jayaraman J and Mathews P M 1971 *J. Math. Phys.* **12** 835–40
Weaver D L, Hammer C L and Good R H Jr 1964 *Phys. Rev.* **135** B241–8